

# A CRITERION FOR $\mathcal{Z}$ -STABILITY WITH APPLICATIONS TO CROSSED PRODUCTS

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**ABSTRACT.** Building on an argument by Toms and Winter, we show that if  $A$  is a simple, separable, unital,  $\mathcal{Z}$ -stable  $C^*$ -algebra, then the crossed product of  $C(X, A)$  by an automorphism is also  $\mathcal{Z}$ -stable, provided that the automorphism induces a minimal homeomorphism on  $X$ . As a consequence, we observe that if  $A$  is nuclear and purely infinite then the crossed product is a Kirchberg algebra.

## 1. INTRODUCTION

In [8], Jiang and Su constructed a  $C^*$ -algebra  $\mathcal{Z}$  (now known as the *Jiang-Su algebra*) which is simple, separable, unital, infinite-dimensional, strongly self-absorbing (in the sense of [18]), nuclear, and has the same Elliott invariant as the complex numbers  $\mathbb{C}$ . A separable  $C^*$ -algebra  $A$  is said to be  $\mathcal{Z}$ -stable if there is an isomorphism  $A \otimes \mathcal{Z} \cong A$ . The property of  $\mathcal{Z}$ -stability appears to be intimately connected to the question of whether or not a simple, separable, nuclear  $C^*$ -algebra is classified by its Elliott invariant. (See [5, 15] for example.)

In this article, we prove the following:

**Theorem 1.1.** Let  $X$  be an infinite compact metric space and let  $A$  be a simple, separable, unital,  $\mathcal{Z}$ -stable  $C^*$ -algebra. Let  $\beta \in \text{Aut}(C(X, A))$ , and suppose that the homeomorphism  $\phi: X \rightarrow X$  induced by  $\beta$  is minimal. Then the crossed product  $C^*$ -algebra,  $C^*(\mathbb{Z}, C(X, A), \beta)$  is also  $\mathcal{Z}$ -stable.

By “the homeomorphism induced by  $\beta$ ,” what is meant is the induced map on the primitive spectrum of  $C(X, A)$  (which can obviously be identified with  $X$ ).

Significant progress has been made in recent years on the classification of crossed product  $C^*$ -algebras arising from minimal dynamical systems. Toms and Winter ([17]) showed that crossed products of infinite, finite-dimensional metric spaces by minimal homeomorphisms have finite nuclear dimension and are  $\mathcal{Z}$ -stable, and consequently that, when the projections in the crossed product separate traces, the crossed products are classified by ordered  $K$ -theory. More recently Elliott and Niu ([4]) have demonstrated that  $\mathcal{Z}$ -stability holds for such crossed products even when  $X$  is infinite-dimensional, so long as the minimal dynamical system has mean dimension zero.

Not as much is known in the case for crossed products of  $C^*$ -algebras of the form  $C(X, A)$ . Hua ([6]) has shown that in the case where  $X$  is the Cantor set,  $A$  has

tracial rank zero, and the automorphisms in the fibre direction are K-theoretically trivial, the resulting crossed product has tracial rank zero. Our result contributes to the understanding of these crossed products, which are further studied by the first-named author in [2]. The main results there assume that  $A$  is locally subhomogeneous, and hence our Corollary 2.2, which deals with the case where  $A$  is purely infinite, is independent of the conclusions in [2].

We would like to thank Andrew Toms and Wilhelm Winter for suggesting the main technical lemma of this paper as a method to prove Theorem 1.1, and for other helpful comments.

## 2. THE PROOF

The proof that certain crossed products here are  $\mathcal{Z}$ -stable is based on an argument of Toms and Winter that crossed products of  $C(X)$  by minimal homeomorphisms are  $\mathcal{Z}$ -stable. This argument appeared (as Theorem 4.4) in a preprint version ([16]) of [17]. In the published version, it was replaced by an indirect proof of this fact.

We have broken apart their argument into a more general criteria for  $\mathcal{Z}$ -stability (the following theorem), followed by an application to our crossed products.

**Lemma 2.1.** Let  $A$  be a separable, unital  $C^*$ -algebra. Define  $c_0, c_{1/2}, c_1 \in C([0, 1])$  by

$$c_0(t) = \begin{cases} 0 & t \leq 3/4, \\ 1 & t = 1, \\ \text{linear} & \text{else.} \end{cases} \quad c_1(t) = \begin{cases} 1 & t = 0, \\ 0 & t \geq 1/4, \\ \text{linear} & \text{else.} \end{cases}$$

$$c_{1/2}(t) = \begin{cases} 0 & t = 0, 1, \\ 1 & 1/4 \leq t \leq 3/4, \\ \text{linear} & \text{else.} \end{cases}$$

Suppose that for any finite set  $\mathcal{F} \subset A$  and any  $\eta > 0$ , there exist  $\mathcal{Z}$ -stable subalgebras  $A_0, A_{1/2}, A_1 \subset A$  and a positive contraction  $h \in A_+$  such that  $A_{1/2} \subseteq A_0 \cap A_1$  and for every  $a \in \mathcal{F}$ , there exist  $a_i \in \overline{c_i(h)A_i c_i(h)}$  for  $i = 0, 1/2, 1$  such that

$$\|a - (a_0 + a_{1/2} + a_1)\| < \eta \text{ and} \\ \|[a_{1/2}, h]\| < \eta$$

It then follows that  $A$  is  $\mathcal{Z}$ -stable.

*Proof.* Using [13, Theorem 7.2.2] and a diagonal sequence argument (cf. [14, Section 4.1]), it suffices to find, for every  $\varepsilon > 0$  and every pair of finite subsets  $\mathcal{F} \subset A$  and  $\mathcal{E} \subset \mathcal{Z}$ , a unital  $*$ -homomorphism

$$\zeta: \mathcal{Z} \rightarrow A_\infty := \prod_{\mathbb{N}} A / \bigoplus_{\mathbb{N}} A,$$

such that  $\|[\iota_A(a), \zeta(z)]\| < \varepsilon$  for all  $a \in \mathcal{F}$  and  $z \in \mathcal{E}$  (with  $\iota: A \rightarrow A_\infty$  the canonical embedding).

Therefore, let  $\varepsilon > 0$  and finite sets  $\mathcal{F} \subset A$  and  $\mathcal{E} \subset \mathcal{Z}$  be given. Define  $d_0, d_{1/2}, d_1 \in C([0, 1])$  by

$$d_0(t) = \begin{cases} 0 & t \leq 1/2, \\ 1 & t \geq 3/4, \\ \text{linear} & \text{else.} \end{cases} \quad d_1(t) = \begin{cases} 1 & t \leq 1/4, \\ 0 & t \geq 1/2, \\ \text{linear} & \text{else.} \end{cases}$$

$$d_{1/2}(t) = \begin{cases} 0 & t = 0 \leq 1/4, t \geq 3/4, \\ 1 & t = 1/2, \\ \text{linear} & \text{else.} \end{cases}$$

Then  $\{c_0, c_{1/2}, c_1\}$  and  $\{d_0, d_{1/2}, d_1\}$  are both partitions of unity for  $[0, 1]$ . We then define

$$C = C^*(d_0 \otimes \mathcal{Z} \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}} \cup d_{1/2} \otimes 1_{\mathcal{Z}} \otimes \mathcal{Z} \otimes 1_{\mathcal{Z}} \cup d_1 \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}} \otimes \mathcal{Z}) \subset C([0, 1]) \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \mathcal{Z}$$

and

$$\tilde{C} = C^*(C([0, 1]) \otimes 1_{\mathcal{Z} \otimes \mathcal{Z} \otimes \mathcal{Z}} \cup C).$$

Identifying  $C^*(d_0, d_{1/2}, d_1)$  in the obvious way with  $C(Y)$  where  $Y = [\frac{1}{4}, \frac{3}{4}]$ , we note that  $C$  is a  $C(Y)$ -algebra, all of whose fibres are isomorphic to  $\mathcal{Z}$ . Therefore, by [3],  $C$  is  $\mathcal{Z}$ -stable, so there exists a unital  $*$ -homomorphism  $\bar{\zeta}: \mathcal{Z} \rightarrow C \subset \tilde{C}$ . Note that

$$\mathcal{S} := (d_0 \otimes \mathcal{Z} \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}}) \cup (d_{1/2} \otimes 1_{\mathcal{Z}} \otimes \mathcal{Z} \otimes 1_{\mathcal{Z}}) \cup (d_1 \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}} \otimes \mathcal{Z}),$$

generates  $C$  as a  $C^*$ -algebra. So, by approximating  $\mathcal{E}$  by  $*$ -polynomials in  $\mathcal{S}$ , we see that there exists  $\beta > 0$  and a finite subset  $\mathcal{S}' \subset \mathcal{S}$  such that, if  $\psi: C \rightarrow B$  is any  $*$ -homomorphism between  $C^*$ -algebras and  $\|[\psi(s), b]\| < \beta$  for all  $s \in \mathcal{S}'$  then  $\|[\psi(\bar{\zeta}(z)), b]\| < \varepsilon/2$  for all  $z \in \mathcal{E}$ .

Set  $M = \max\{\|z\| : z \in \mathcal{E}\}$  and let  $\eta \leq \varepsilon/(2M)$  be sufficiently small so that if  $\| [a_{1/2}, h] \| < \eta$  then  $\| [a_{1/2}, d_i(h)] \| < \beta$  for  $i = 0, 1$ . Use the hypothesis to find the subalgebras  $A_i$  and the positive contraction  $h$ .

Since each  $A_i$  is  $\mathcal{Z}$ -stable, there exists unital  $*$ -homomorphisms  $\bar{\rho}_i: \mathcal{Z} \rightarrow (A_i)_\infty \cap \iota_{A_i}(A_i) \subseteq A_\infty \cap \iota_A(A_i)$ . Having found  $\bar{\rho}_{1/2}$  first, a speeding up argument (cf. the proof of [19, Proposition 4.4]) shows that we can arrange that  $\bar{\rho}_i(\mathcal{Z})$  commutes with  $\bar{\rho}_{1/2}(\mathcal{Z})$ , for  $i = 0, 1$ .

We may define a unital  $*$ -homomorphism  $\gamma: \tilde{C} \rightarrow A_\infty$  by setting

$$\begin{aligned} \gamma(fd_0 \otimes z \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}}) &= (fd_0)(h)\bar{\rho}_0(z), \\ \gamma(fd_{1/2} \otimes 1_{\mathcal{Z}} \otimes z \otimes 1_{\mathcal{Z}}) &= (fd_{1/2})(h)\bar{\rho}_{1/2}(z), \\ \gamma(fd_1 \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}} \otimes z) &= (fd_1)(h)\bar{\rho}_1(z), \end{aligned}$$

for all  $f \in C([0, 1])$ . (The proof that this defines a  $*$ -homomorphism mainly consists of checking that anything occurring on the right-hand sides of two different equations above commutes.) Finally, define  $\zeta = \gamma \circ \bar{\zeta}: \mathcal{Z} \rightarrow A_\infty$ .

For  $a \in \mathcal{F}_1$ , let  $a \approx_\varepsilon a_0 + a_{1/2} + a_1$  as in the hypothesis. In fact, we may assume that  $a_i = c_i(h)a'_i c_i(h)$  exactly, for some  $a'_i \in A_i$ . Then, for  $z \in \mathcal{E}$ ,

$$[a, \zeta(z)] \approx_{2\|z\|\varepsilon} [a_0, z] + [a_{1/2}, z] + [a_1, z].$$

Notice that since  $\bar{\zeta}(z) \in \tilde{C}$ , it follows that there exists  $z_0 \in \mathcal{Z}$  such that

$$\bar{\zeta}(z)(t) = z_0 \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}}$$

for all  $t \in [0, 1/4]$ . Consequently,

$$\begin{aligned}\zeta(z)a_1 &= \gamma(c_1 \otimes z_0 \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}})a'_1c_1(h) \\ &= c_1(h)\bar{\rho}_1(z_0)a'_1c_1(h) \\ &= c_1(h)a'_1\bar{\rho}_1(z_0)c_1(h) \\ &= a_1\zeta(z).\end{aligned}$$

(the last step is essentially done by reversing earlier steps). Likewise,  $\zeta(z)a_0 = a_0\zeta(z)$ . Also, we have for  $z \in \mathcal{Z}$ ,

$$\begin{aligned}[a_{1/2}, \gamma(d_0 \otimes z \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}})] &= [a_{1/2}, d_0(h)\bar{\rho}_0(z)] \\ &= [a_{1/2}, d_0(h)] \\ &< \beta.\end{aligned}$$

Likewise, we find that  $\|[a_{1/2}, \gamma(s)]\| < \beta$  for all  $s \in \mathcal{S}'$ , and therefore,

$$\|[a_{1/2}, \gamma(z)]\| < \eta/2$$

for  $z \in \mathcal{E}$ . It follows that

$$\|[a, \zeta(z)]\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which completes the proof.  $\square$

*Proof of Theorem 1.1.* Set  $B := C^*(\mathbb{Z}, C(X, A), \beta)$ , and let  $u \in B$  denote the canonical unitary. We shall show that  $B$  satisfies the hypotheses of Lemma 2.1. Let  $\eta > 0$  and a finite set  $\mathcal{F} \subset B$  be given. First, we may assume that

$$\mathcal{F} \subset \{u^j C(X) : 0 \leq j \leq k-1\}$$

for some  $k > 1$ , since the linear span of these elements and their adjoints is dense in  $B$ . Set  $M := \max\{\|f\| : u^j f \in \mathcal{F}\}$ . Combining Propositions 1.1 and 3.2 of [17], there exists  $h \in C(X) \otimes 1_A$  and points  $x_0, x_1 \in X$  with disjoint orbits such that  $h(x_j) = j$  (for  $j = 0, 1$ ) and such that  $h, u$  satisfy the relations

$$\|[u, h]\| < \frac{\eta}{3M}, \quad \|[u, c_i(h)^{1/k}]\| < \frac{2\eta}{3Mk(k-1)}$$

for  $i = 0, 1/2, 1$  and with the  $c_i$  given as in the statement of Proposition 2.1. It then follows that, for  $a = u^\ell f \in \mathcal{F}$  and  $i = 0, 1/2, 1$ , we have

$$\begin{aligned}\|c_i(h)a - c_i(h)^{(k-\ell)/k} f(c_i(h)^{1/k} u)^\ell\| &\leq M\|c_i(h)^{\ell/k} u^\ell - (c_i(h)^{1/k} u)^\ell\| \\ &\leq M\|[u, c_i(h)^{1/k}]\|(\ell + (\ell-1) + \cdots + 1) \\ &< \frac{M\ell(\ell+1)}{2} \frac{2\eta}{3Mk(k-1)} \\ &\leq \eta/3.\end{aligned}$$

Thus,

$$a = c_0(h)a + c_{1/2}(h)a + c_1(h)a \approx_\eta a_0 + a_{1/2} + a_1,$$

where  $a_i = c_i(h)^{(k-\ell)/k} f(c_i(h)^{1/k} u)^\ell$ .

Set  $Y_i := \{x_i\}$  for  $i = 0, 1$  and  $Y_{1/2} := \{x_0, x_1\}$ . For  $i = 0, 1/2, 1$ , set

$$A_i := C^*(C(X, A) \cup uC_0(X \setminus Y_i, A)) \subseteq B.$$

We see that  $A_{1/2} \subseteq A_0 \cap A_1$ , and  $a_i \in \overline{c_i(h)A_i c_i(h)}$ . Results of [2] show that each  $A_i$  is  $\mathcal{Z}$ -stable. Moreover, for  $i = 0, 1/2, 1$ , (and in particular, for  $i = 1/2$ ),

$$\begin{aligned} \|[a_{1/2}, h]\| &\leq \|[c_i(h)u^\ell f, h]\| + \frac{2\eta}{3} \\ &\leq M \|[u^\ell, h]\| + \frac{2\eta}{3} \\ &< \frac{M\eta}{3M} + \frac{2\eta}{3} \\ &= \eta. \end{aligned}$$

This verifies the hypotheses of Lemma 2.1, and therefore,  $B$  is  $\mathcal{Z}$ -stable.  $\square$

It is well-known (see [12]) that exact  $\mathcal{Z}$ -stable  $C^*$ -algebras are either stably finite or purely infinite. This allows us to obtain a useful corollary in the case where  $A$  is nuclear and purely infinite.

**Corollary 2.2.** Adopt the hypotheses and notation of Theorem 1.1 and assume in addition that  $A$  is nuclear and purely infinite. Then  $B$  is a Kirchberg algebra. Consequently,  $B$  has nuclear dimension at most 3.

*Proof.* By Theorem 1.1, the algebra  $B$  is  $\mathcal{Z}$ -stable. It is clearly infinite, since it contains the purely infinite algebra  $A$  as the subalgebra  $1_{C(X)} \otimes A$ , and it is nuclear. Therefore it is purely infinite. Since  $B$  is simple (this essentially follows from [1]), it is a Kirchberg algebra. The conclusion about the nuclear dimension of  $B$  follows immediately from Theorem 7.1 of [10].  $\square$

If  $A$  is in the UCT class, then so is the algebra  $B$  of Corollary 2.2, and hence such algebras are classified by their K-theory using the theorems of Kirchberg and Phillips ([7], [11]).

## REFERENCES

- [1] R. J. Archbold and J. S. Spielberg, *Topologically free actions and ideals in discrete  $C^*$ -dynamical systems*, Proceedings of the Edinburgh Mathematical Society (Series 2), **37**(1994), 119–124.
- [2] J. Buck, *Large subalgebras of certain crossed product  $C^*$ -algebras*, in preparation.
- [3] M. Dadarlat and W. Winter, *Trivialization of  $C(X)$ -algebras with strongly self-absorbing fibres*, Bull. Soc. Math. France **136**(2008), no. 4, 575–606.
- [4] G. A. Elliott and Z. Niu, *The  $C^*$ -algebra of a minimal homeomorphism of zero mean dimension*, preprint (arXiv:1406.2382 [math.OA]).
- [5] G. A. Elliott and A. S. Toms, *Regularity properties in the classification program for separable amenable  $C^*$ -algebras*, Bull. Amer. Math. Soc. (N.S.) **45**(2008), no. 2, 229–245.
- [6] J. Hua, *Crossed products by  $\alpha$ -simple automorphisms on  $C^*$ -algebras  $C(X, A)$* , preprint (arXiv:0910.3299v2 [math.OA]).
- [7] E. Kirchberg, *The classification of purely infinite  $C^*$ -algebras using Kasparov's theory*, preprint, 1994.
- [8] X. Jiang and H. Su, *On a simple unital projectionless  $C^*$ -algebra*, Amer. J. Math. **121**(1999), 359–413.
- [9] E. Kirchberg and M. Rørdam, *Non-simple purely infinite  $C^*$ -algebras*, Amer. J. Math. **122**(2000), 637–666.
- [10] H. Matui and Y. Sato, *Decomposition rank of UHF-absorbing  $C^*$ -algebras*, preprint (arXiv:1303.4371v2 [math.OA]).
- [11] N. C. Phillips, *A classification theorem for nuclear purely infinite simple  $C^*$ -algebras*, Documenta Math. (5)(2000), 49–114.
- [12] M. Rørdam, *The stable and the real rank of  $\mathcal{Z}$ -absorbing  $C^*$ -algebras*, Int. J. Math. **15**(2004), 1065–1084.

- [13] M. R. Rørdam and E. Størmer, *Classification of nuclear  $C^*$ -algebras. Entropy in operator algebras*, volume 126 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Operator Algebras and Non-commutative Geometry, 7.
- [14] A. Tikuisis, *Nuclear dimension,  $\mathcal{Z}$ -stability, and algebraic simplicity for stably projectionless  $C^*$ -algebras*, Math. Ann., to appear.
- [15] A. S. Toms, *On the classification problem for nuclear  $C^*$ -algebras*, Ann. of Math. (2) **167**(2008), 1059–1074.
- [16] A. S. Toms and W. Winter, *Minimal dynamics and  $K$ -theoretic rigidity: Elliott’s conjecture*, preprint (arXiv:math.OA/0903.4133v1).
- [17] A. S. Toms and W. Winter, *Minimal dynamics and  $K$ -theoretic rigidity: Elliott’s conjecture*, Geom. and Func. Anal. (1) **23** (2013), 467–481.
- [18] A. S. Toms and W. Winter, *Strongly self-absorbing  $C^*$ -algebras*, Trans. Amer. Math. Soc. **359**(2007), 3999–4029.
- [19] W. Winter, *Nuclear dimension and  $\mathcal{Z}$ -stability of pure  $C^*$ -algebras*, Invent. Math **187**(2012) no. 2, 259–342